CALCULATION OF THE TEMPERATURE FIELD ASSOCIATED WITH THE INJECTION OF A HEAT-TRANSFER AGENT INTO AN OIL STRATUM IN THE LINEAR CASE

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Equations are obtained for determining the temperature field of an oil stratum and the surrounding rock when a heat-transfer agent is pumped into the stratum. These equations are derived on the basis of a boundary condition of the third kind at the point where the fluid enters the bed, which corresponds more closely to the actual pumping conditions than the usual boundary condition of the first kind.

One of the most important problems relating to petroleum extraction is the question of the temperature field of the oil stratum and the surrounding rock when a fluid, whose temperature is different from the initial local temperature, is injected into the stratum. We propose to consider the following problem. Let an incompressible fluid, whose temperature is different from the initial local temperature, be pumped into an infinitely deep stratum through a straight shaft located in the plane $-\infty < \bar{y} < +\infty$, $x = 0$, $-h \le \bar{z} \le 0$. It is required to determine the temperature of the stratum in the region $0 < \bar{x} < +\infty$, $-\infty < \bar{y} +\infty$, $-h \le \bar{z} \le 0$ and the surrounding rock in the regions $0 < \bar{x}, \bar{y}, \bar{z} < +\infty$ and $0 < \bar{x}, \bar{y} < +\infty$, $-\infty < \bar{z} < -h$. We make the following assumptions [1-3].

1) The rocks above and below the stratum have the same thermophysical properties and are impermeable for the fluid.

2) Heat transfer between the fluid and the structure of the stratum is instantaneous.

3) The rocks surrounding the stratum are assumed to be thermally anisotropic: their thermal conductivity in the direction of the $\bar{z}$ axis is considered to be equal to their true thermal conductivity and that in the direction of the $\bar{x}$ axis equal to zero.

With respect to the thermal conductivity of the stratum in the direction of the $\bar{x}$ axis we will consider two possibilities.

I. The thermal conductivity of the stratum in the direction of the $\bar{x}$ axis is equal to the true thermal conductivity of the stratum (incomplete lumped-capacitance model described in [1, 2]).

II. The thermal conductivity of the stratum in the direction of the $\bar{x}$ axis is equal to zero (Lauwerier's model [3]).

At the point where the fluid enters the stratum it is usual to formulate a boundary condition of the first kind [1-3]

$$u\big|_{\substack{x=0\\z=0}} = 1, \tag{1}$$

according to which the temperature of the stratum in the plane x = 0 is instantaneously equal to the temperature of the pumped fluid. This does not correspond to actual conditions. In fact, if one begins to pump into a porous wall with initial temperature T_0 a fluid whose temperature at a certain distance from the wall is maintained equal to $T_f \ne T_0$, then the wall surface temperature does not become instantaneously equal to T_f. In [4] it was shown that in the linear case condition (1) is only approximately correct at sufficiently

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large values of the nondimensional time t or the convective parameter 2γ and that the actual pumping process corresponds to the boundary condition of the third kind employed below. With a boundary condition of the third kind the temperature of the stratum in the plane $\bar{x} = 0$ only gradually (theoretically as $t \to \infty$) approaches the temperature of the pumped fluid. In [4] a boundary condition of the third kind was used to prove the coincidence of the integral heat losses in the linear and radial cases. Below, this condition is used to determine the temperature field of the stratum and the surrounding rocks.

In the linear case the mathematical formulation of the problem in nondimensional quantities for models I and II is as follows

$$\frac{\partial^2 u}{\partial z^2} = \frac{\partial u}{\partial t}, \quad 0 < x, \ z, \ t < +\infty, \tag{2}$$

$$a^2 \frac{\partial^2 u}{\partial x^2} - 2\gamma \frac{\partial u}{\partial x} + \mu \frac{\partial u}{\partial z} = \frac{\partial u}{\partial t}, \quad z = 0, \quad 0 < x, \quad t < +\infty, \tag{3}$$

$$\left(\frac{\partial u}{\partial x} - Hu\right)\Bigg|_{\substack{x=0\\ z=0}} = -H, \tag{4}$$

$$u\big|_{t=0} = 0; \quad \lim_{\sqrt{x^2+z^2}\to\infty} u = 0. \tag{5}$$

In the case of model I $a^2 = a_T^2/a_S^2$, in the case of model II $a^2 = 0$. We apply a Laplace–Carson transformation to problem (2)-(5), setting

$$\bar{u}(x, z, p) = p\int_0^\infty u(x, z, t)\exp(-pt)\,dt. \tag{6}$$

Solving the equation for the transform, we obtain

$$\bar{u}(x, z, \mu^2 p) \div \int_0^t \varphi(\tau_1)\,d\tau_1 \int_0^{t-\tau_1} \frac{\tau_1 \exp\left(-\frac{\tau_1^2}{4v}\right)}{2\sqrt{\pi}\, v^{\frac{3}{2}}} \operatorname{erfc}\left(\frac{z\mu}{2\sqrt{t-\tau_1-v}}\right) dv, \tag{7}$$

where

$$\varphi(t) = \frac{\exp(-ct)}{2\mu}\left\{\left(a\sqrt{g-c} - \frac{f}{\mu}\right)\exp\left(-\frac{x\mu}{a}\sqrt{g-c}\right)\right.$$
$$\times \operatorname{erfc}\left[\frac{x\mu}{2a\sqrt{t}} - \sqrt{(g-c)t}\right] - \left(a\sqrt{g-c} + \frac{f}{\mu}\right)\exp\left(\frac{x\mu}{a}\sqrt{g-c}\right)$$
$$\left.\times \operatorname{erfc}\left[\frac{x\mu}{2a\sqrt{t}} + \sqrt{(g-c)t}\right]\right\} + \frac{a}{\mu\sqrt{\pi t}}\exp\left(-gt - \frac{x^2\mu^2}{4a^2 t}\right), \tag{8}$$

$$c = \frac{2\gamma H - a^2H^2}{\mu^2}, \quad g = \frac{\gamma^2}{\mu^2 a^2}, \quad f = a^2 H - \gamma. \tag{9}$$

Evaluating the inner integral in (7) by applying a Laplace transformation with respect to the variable $t - \tau_1$, we finally obtain

$$u(x, z, t) = H\exp\left(\frac{\gamma x}{a^2}\right)\int_0^{\mu^2 t} \varphi(\tau_1)\operatorname{erfc}\left(\frac{z\mu + \tau_1)}{2\sqrt{\mu^2 t - \tau_1}}\right)d\tau_1. \tag{10}$$

The function (10) is the solution of problem (2)-(5). We will now consider models I and II separately.

<u>I. Incomplete Lumped-Capacitance Model.</u> In (10) we set

$$a^2 = \frac{a_S^2}{a_T^2}. \tag{11}$$

After certain transformation this gives

$$u(x,\ z,\ t) = 2Ht \int_0^1 \left\{ \frac{a}{\sqrt{\pi t}} \exp\left[-\left(\frac{\gamma s}{a} \sqrt{t} - \frac{x}{2as\sqrt{t}} \right)^2 \right] \right.$$

$$\left. - \gamma s \exp\left(\frac{2\gamma x}{a^2} \right) \operatorname{erfc}\left(\frac{\gamma s\sqrt{t}}{a} + \frac{x}{2as\sqrt{t}} \right) \right\} \operatorname{erfc}\left(\frac{z + \mu^2 s\sqrt{t}}{2\sqrt{1-s^2}} \right) ds. \tag{12}$$

Equation (12) determines the stratum roof temperature.

In order to obtain the stratum temperature it is sufficient to set z = 0 in (12)

$$u_s(x,\ t) = u(x,\ 0,\ t) = 2Ht \int_0^1 \left\{ \frac{a}{\sqrt{\pi t}} \exp\left[-\left(\frac{\gamma s}{a} \sqrt{t} - \frac{x}{2as\sqrt{t}} \right)^2 \right] \right.$$

$$\left. - \gamma s \exp\left(\frac{2\gamma x}{a^2} \right) \operatorname{erfc}\left(\frac{\gamma s\sqrt{t}}{a} + \frac{x}{2as\sqrt{t}} \right) \right\} \operatorname{erfc}\left(\frac{\mu^2 s\sqrt{t}}{2\sqrt{1-s^2}} \right) ds. \tag{13}$$

In particular, the temperature at the point where the fluid enters the stratum is equal to

$$u_s(0,\ t) = 2Ht \int_0^1 \left\{ \frac{a}{\sqrt{\pi t}} \exp\left(-\frac{\gamma^2 t s^2}{a^2} \right) - \gamma s \operatorname{erfc} \frac{\gamma s\sqrt{t}}{a} \right\} \operatorname{erfc}\left(\frac{\mu^2 s\sqrt{t}}{2\sqrt{1-s^2}} \right) ds. \tag{14}$$

Equation (13) gives the maximum error introduced by using a boundary condition of the first kind (1) instead of a boundary condition of the third kind (4). This error may be of the order of unity, since from (14) it follows that

$$\lim_{t\to+0} u_s(0,\ t) = 0; \qquad \lim_{\substack{Q\to+0 \\ t>0}} u_s(0,\ t) = 0. \tag{15}$$

Accordingly, in the linear case it is preferable to employ Eqs. (12) and (13) for calculating the temperature field of the stratum and the surrounding rock rather than the equations obtained in [1] with boundary conditions of the first kind (1).

II. Lauwerier Model. Passing to the limit in (10) as $a^2 \to +0$, making the substitution $t - \tau/\mu^2 = \xi$, and integrating by parts, we obtain

$$u(x,\ z,\ t) = 2\gamma H \exp(Hx - 2\gamma Ht) \left\{ \frac{1}{2\gamma H} \exp\left[2\gamma H \left(t - \frac{x}{2\gamma} \right) \right] \right.$$

$$\times \operatorname{erfc}\left(\frac{z + \dfrac{\mu x}{2\gamma}}{2\sqrt{t - \dfrac{x}{2\gamma}}} \right) - \frac{1}{2\gamma H\sqrt{\pi}} \exp\left[\frac{\mu(z + \mu t)}{2} \right]$$

$$\times \left[\frac{\mu}{2} \int_0^{t-\frac{x}{2\gamma}} \exp\left(-\frac{\mu^2 - 8\gamma H}{4} \xi - \frac{z+\mu t}{4\xi} \right) \frac{d\xi}{\sqrt{\xi}} \right.$$

$$\left. \left. - \frac{z+\mu t}{2} \int_0^{t-\frac{x}{2\gamma}} \exp\left(-\frac{\mu^2 - 8\gamma H}{4} \xi - \frac{z+\mu t}{4\xi} \right) \frac{d\xi}{\xi\sqrt{\xi}} \right] \right\} \eta\left(t - \frac{x}{2\gamma} \right), \tag{16}$$

where

$$\eta\left(t - \frac{x}{2\gamma} \right) = \begin{cases} 0, & t < \dfrac{x}{2\gamma}, \\ 1, & t > \dfrac{x}{2\gamma}. \end{cases}$$

In order to evaluate the integrals in (16) we employ the formula*

$$\int_0^t \exp\left(-\alpha\tau-\frac{\beta}{\tau}\right)\frac{d\tau}{\tau^{n+\frac{1}{2}}}=\frac{d^n}{d\beta^n}\cdot\frac{\sqrt{\pi}}{2\alpha}\left[\exp(-2\sqrt{\alpha\beta})\right.$$

$$\left.\times\operatorname{erfc}\left(\sqrt{\frac{\beta}{t}}-\sqrt{\alpha t}\right)-\exp(2\sqrt{\alpha\beta})\operatorname{erfc}\left(\sqrt{\frac{\beta}{t}}+\sqrt{\alpha t}\right)\right]$$

$$(n=0,\ 1,\ 2,\ \ldots). \tag{17}$$

Then (16) takes the form

$$u(x,z,t)=\left\{\operatorname{erfc}\frac{z+\frac{\mu x}{2\gamma}}{2\sqrt{t-\frac{x}{2\gamma}}}-\frac{1}{2}\exp\left[\frac{\mu(z+\mu t)}{2}-2\gamma H\left(t-\frac{x}{2\gamma}\right)\right]\right.$$

$$\times\left[\left(\frac{\mu}{\sqrt{\mu^2-8\gamma H}}+1\right)\exp\left(-\frac{z+\mu t}{2}\sqrt{\mu^2-8\gamma H}\right)\operatorname{erfc}\left(\frac{z+\mu t}{2\sqrt{t-\frac{x}{2\gamma}}}\right.\right.$$

$$\left.-\sqrt{(\mu^2-8\gamma H)\left(t-\frac{x}{2\gamma}\right)}\right)-\left(\frac{\mu}{\sqrt{\mu^2-8\gamma H}}-1\right)\exp\left(\frac{z+\mu t}{2}\sqrt{\mu^2-8\gamma H}\right)$$

$$\left.\left.\times\operatorname{erfc}\left(\frac{z+\mu t}{2\sqrt{t-\frac{x}{2\gamma}}}+\sqrt{(\mu^2-8\gamma H)\left(t-\frac{x}{2\gamma}\right)}\right)\right]\right\}\eta\left(t-\frac{x}{2\gamma}\right). \tag{18}$$

The function u(x, z, t), given by Eq. (18), is the solution of problem (2)-(5) at $a^2 = 0$, i.e., an extension of the Lauwerier equation [3] to the case of a boundary condition of the third kind. The first term in (18) coincides with Lauwerier's expression [3], which does not depend on the thermal conductivity of the stratum λ_s. The remaining terms of Eq. (18) take the thermal conductivity of the stratum partially into account. If in (18) we set $\lambda_s = 0$ (i.e., $H \to \infty$), then all the terms in (18), apart from the first, will be equal to zero and Eq. (18) goes over into the Lauwerier expression [3]. As before, relation (16) holds for Eq. (18), and therefore the error introduced by using Lauwerier's expression [3] instead of (18) may be of the order of unity.

If in (18) $\mu^2 < 8\gamma H$, then one must make the transformation

$$\operatorname{erfc}(\alpha+i\beta)=1-\frac{2}{\sqrt{\pi}}\int_0^{\alpha+i\beta}\exp(-\xi^2)\,d\xi=1-\frac{2i}{\sqrt{\pi}}\int_0^{\beta-\alpha i}\exp(y^2)\,dy. \tag{19}$$

The function $\int_0^z \exp(y^2)dy$ is tabulated in [6].

NOTATION

$u(x, z, t) = [T(x, z, t) - T_0]/(T_f - T_0)$	is the nondimensional roof temperature;
$u_s(x, t) = u(x, 0, t)$	is the nondimensional stratum temperature;
$T(x, z, t)$	is the dimensional roof temperature;
T_0	is the initial temperature of the stratum and the surrounding rocks;
$T_f = \text{const}$	is the fluid temperature at a certain distance from the entrance to the stratum;
$x = 2\bar{x}/h$, $z = 2\bar{z}/h$, $t = 4a_r^2\bar{t}/h^2$	are the nondimensional coordinates and time, respectively;
h	is the thickness of the stratum;
a_s^2, a_r^2, λ_s, λ_r	are the thermal diffusivities and thermal conductivities of the stratum and the roof, respectively;

* For brevity the derivation of Eq. (17) has been omitted.

c_s, c_r, c_f	are the volume specific heats of the stratum, the roof, and the fluid;
$a^2 = a_s^2/a_r^2$	is the case of model I, $a^2 = 0$ in the case of model II;
$2\gamma = Qc_f/2a_r^2c_s$	is the convective parameter;
Q	is the volume flow rate of injected fluid per linear meter of the shaft;
$\mu = c_r/c_s$;	
$H = Qc_f/2\lambda_s$.	

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